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RESEARCH ARTICLE

Airy criterion and the ideal mapping of an ellipsoid of revolution onto a sphere

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Abstract: This paper addresses the problem of constructing an ideal projection of an ellipsoid of revolution onto a sphere based on the Airy criterion. General equations required to solve the problem are derived. In particular, the Euler–Urmaev system is obtained, allowing a clear illustration of Gauss's theorem that a distortion-free projection between these surfaces cannot exist. The Euler–Ostrogradsky system is also derived to find the projection that minimizes distortion according to the Airy criterion. Natural boundary conditions for the ideal projection are analyzed. It is shown that on the boundary of the mapping region, Tissot's indicatrices are aligned either along the normals or tangents to the boundary, and one of the extremal linear scale factors is equal to unity. Since the value of the Airy criterion depends not only on the projection's mapping functions but also on the radius of the sphere, an additional integral condition is introduced alongside the Euler–Urmaev system, the Euler–Ostrogradsky system, and the natural boundary conditions. According to this condition, the integral of the area distortion over the entire mapping region of the ideal projection must be equal to zero. Two specific cases are examined in detail: projection of the entire ellipsoid and of a region bounded by a parallel. For comparison, conformal projections optimized according to the Airy criterion were also constructed for the same mapping regions. The resulting ideal projections can be used in geodesy for solving direct and inverse geodetic problems, and in cartography for constructing double projections.

Keywords: Airy criterion, ideal map projection, Euler–Urmaev equations, area distortion, conformal mapping, Tissot's indicatrix, double projection

1 Introduction

When comparing projections of the ellipsoid onto a plane and onto a sphere, it becomes evident that the latter have been significantly less studied, both in cartography and geodesy. This is likely due to the fact that projections of the ellipsoid onto a sphere often serve as an intermediate step in various applied tasks.

In cartography, for example, when using double projections [6, p. 257]; [3, p. 41]; [13], the surface of the ellipsoid is first projected onto a sphere, and the resulting image is then projected onto a plane. The final outcome is a planar projection, while the spherical surface is used merely as an auxiliary model to simplify the transformation process: projections from a sphere to a plane are considerably simpler than direct projections from an ellipsoid.

A similar approach is common in geodesy. When solving various problems on the ellipsoidal surface—such as the direct and inverse geodetic problems, or the adjustment of geodetic networks—geodetic data are often projected onto a sphere using methods such as those of Bessel or Gauss [2, 9, 10, 14]. The corresponding computations are then carried out on the sphere, and the results are subsequently projected back onto the ellipsoid using the same methods. This approach is justified, since geodetic problems are solved on the sphere using significantly simpler formulas than on the ellipsoid.

It should be noted that conformal projections of the ellipsoid onto the sphere have seen the widest application, primarily due to the fundamental work of Gauss [5]. A comprehensive historical review of the development of the theory of conformal mappings from the ellipsoidal surface to the sphere is provided by Lapaine [12].

Since distortions introduced by projections from the ellipsoid to the sphere are significantly smaller than those from the sphere to the plane, the ellipsoid-to-sphere projection is generally chosen to ensure maximum simplicity of the projection formulas. As a result, the problem of finding an ideal projection—i.e., the best possible projection from the ellipsoid to the sphere—has not yet been explicitly formulated or addressed.

In [16, pp. 55–63], a solution is presented for constructing an ideal projection of an arbitrary regular surface onto a plane, based on the Airy criterion. It is shown that the solution relies on the following system of differential equations:

$$\begin{aligned} \frac{\partial}{\partial q}[r(a+b-1)\cos\beta] &= -\frac{\partial}{\partial\lambda}[r(a+b-1)\sin\beta]\,, \\ \frac{\partial}{\partial q}[r(a+b-1)\sin\beta] &= \frac{\partial}{\partial\lambda}[r(a+b-1)\cos\beta] \end{aligned} \tag{1}$$

Here, q and λ denote the isometric latitude and longitude of a point on the ellipsoid, respectively; a and b are the maximum and minimum linear scale factors of the projection; β is the angle representing the difference between the azimuth of the principal direction on the ellipsoid and the grid azimuth of the same direction on the plane; r is the metric coefficient of the ellipsoidal surface. As shown in [16, p. 35], this function depends on the choice of the coordinate system on the surface. For the coordinate system defined by isometric latitude and longitude, the metric coefficient is equal to the radius of the parallel. For isometric coordinates on the surface of a sphere, corresponding to the coordinates of the polar system (zenith distance and azimuth), the metric coefficient is equal to the radius of the almucantar, which differs from the radius of the parallel.

According to equation (1), for Airy-ideal projections of the ellipsoid onto the plane, the functions $\ln[r(a+b-1)]$ and $(-\beta)$ form a pair of conjugate harmonic functions.

Moreover, it has been shown that the Tissot's indicatrices are oriented either along the normals or the tangents along the boundary of the mapping region, and one of the extremal scale factors is equal to unity [16, p. 57].

In this study, the problem of constructing an ideal projection of an ellipsoid of revolution onto a sphere is formulated. General equations for solving this problem are derived. In particular, the Euler–Urmaev system is obtained, which makes it possible to clearly demonstrate the validity of Gauss's theorem [6, p. 81] on the impossibility of a distortion-free projection between these surfaces. In addition, the Euler–Ostrogradsky system of equations is derived, the solution of which allows for the determination of the ideal projection of the ellipsoid onto the sphere.

The natural boundary conditions that must be satisfied by an ideal ellipsoid-to-sphere projection are analyzed in detail. Two special cases are considered: (1) the ideal projection of the entire ellipsoidal surface onto the sphere, and (2) the ideal projection of a portion of the ellipsoid bounded by a parallel. The resulting solutions are compared with the corresponding solutions for projections of the ellipsoid onto a plane.

2 Euler-Urmaev equations for projections of an ellipsoid of revolution onto a sphere

According to [3, pp. 31, 16], the local linear scale factors along meridians and parallels, as well as the sine of the angle between the images of the meridians and parallels in the projection of the ellipsoid of revolution onto the sphere, are determined using the following formulas:

$$m = \frac{R\cos\Phi}{r}\sqrt{e}, \qquad n = \frac{R\cos\Phi}{r}\sqrt{g}, \qquad \sin i = \frac{h}{\sqrt{e\,g}}$$

Applying the well-known theorems of Apollonius [3, p. 19] to these formulas, and performing standard algebraic transformations, yields the following expressions for the extremal linear scale factors:

$$a^2 = \frac{R^2\cos^2\Phi}{2r^2}\left(e + g + \sqrt{(e+g)^2 - 4h^2}\right) \tag{2}$$

$$b^2 = \frac{R^2\cos^2\Phi}{2r^2}\left(e + g - \sqrt{(e+g)^2 - 4h^2}\right) \tag{3}$$

where R is the radius of the sphere; Φ is the geographic latitude of a point on the sphere; e, g are the coefficients of the first fundamental form defined in the transformation from the original surface to the sphere; h is the Jacobian of the mapping, respectively equal to:

$$\begin{aligned} e &= Q_q^2 + \Lambda_q^2, \\ g &= Q_\lambda^2 + \Lambda_\lambda^2, \\ h &= Q_q\,\Lambda_\lambda - Q_\lambda\,\Lambda_q \end{aligned} \tag{4}$$

In these equations, q, λ are the isometric coordinates on the ellipsoid; Q, Λ are the isometric coordinates of the point mapped from the ellipsoid onto the sphere, as well as the mapping functions when projecting points of the ellipsoid onto the sphere; $Q_q, Q_\lambda, \Lambda_q, \Lambda_\lambda$

are the partial derivatives of the mapping functions Q, Λ with respect to the isometric coordinates q, λ on the ellipsoid.

The product of equations (2) and (3), after algebraic manipulation, leads to the following expression:

$$2ab = 2\frac{hR^2 \cos^2 \Phi}{r^2} \tag{5}$$

By adding equations (2) and (3), and taking into account equation (5), then subtracting equation (2) from equation (3), the following expressions are obtained:

$$\begin{aligned} \frac{r(a+b)}{R\cos\Phi} &= \sqrt{(Q_q + \Lambda_\lambda)^2 + (Q_\lambda - \Lambda_q)^2}, \\ \frac{r(a-b)}{R\cos\Phi} &= \sqrt{(Q_q - \Lambda_\lambda)^2 + (Q_\lambda + \Lambda_q)^2} \end{aligned} \tag{6}$$

Introducing the notations

$$\tan\beta = \frac{Q_\lambda - \Lambda_q}{Q_q + \Lambda_\lambda}, \qquad \tan\chi \ = \frac{Q_\lambda + \Lambda_q}{Q_q - \Lambda_\lambda} \tag{7}$$

yields

$$\begin{aligned} Q_q + \Lambda_\lambda &= \frac{r(a+b)}{R\cos\Phi}\cos\beta, \quad Q_\lambda - \Lambda_q = \frac{r(a+b)}{R\cos\Phi}\sin\beta, \\ Q_q - \Lambda_\lambda &= \frac{r(a-b)}{R\cos\Phi}\cos\chi, \quad Q_\lambda + \Lambda_q = \frac{r(a-b)}{R\cos\Phi}\sin\chi \end{aligned} \tag{8}$$

Despite the fact that the angle β in equations (7) is defined for the projection of the ellipsoid onto the sphere, whereas the angle β in equations (1) refers to the projection of the ellipsoid onto the plane, the symbol β represents the same geometric quantity: the rotation angle of the local coordinate frame (meridional–parallel axes) under the mapping. The analytical expressions differ because they involve different pairs of differential coordinate components (ellipsoid–plane versus ellipsoid–sphere), but the definition of β as the orientation parameter of the local mapping is identical. Therefore, in this paper, the notation β is retained to emphasize that both transformations share a common underlying differential–geometric structure.

Formulas (8) allow us to obtain the relationship between the partial derivatives of the mapping functions Q, Λ and the extremal scale factors of the projection, namely:

$$\begin{aligned} Q_q &= \frac{r}{2R\cos\Phi}\left[(a+b)\cos\beta + (a-b)\cos\chi\right], \\ Q_\lambda &= \frac{r}{2R\cos\Phi}\left[(a+b)\sin\beta + (a-b)\sin\chi\right], \\ \Lambda_q &= -\frac{r}{2R\cos\Phi}\left[(a+b)\sin\beta - (a-b)\sin\chi\right], \\ \Lambda_\lambda &= \frac{r}{2R\cos\Phi}\left[(a+b)\cos\beta - (a-b)\cos\chi\right] \end{aligned} \tag{9}$$

For the mapping functions to be twice continuously differentiable over the mapping region, their partial derivatives must satisfy the conditions:

$$\begin{aligned} Q_{q\lambda} &= Q_{\lambda q}, \\ \Lambda_{q\lambda} &= \Lambda_{\lambda q} \end{aligned} \tag{10}$$

This means that the four projection parameters a, b, β, and χ cannot be chosen arbitrarily. They must satisfy the following system of equations:

$$\begin{aligned} \frac{\partial}{\partial\lambda}\Big(\frac{r}{R\cos\Phi}[(a+b)\cos\beta+(a-b)\cos\chi]\Big) &= \frac{\partial}{\partial q}\Big(\frac{r}{R\cos\Phi}[(a+b)\sin\beta+(a-b)\sin\chi]\Big), \\ -\frac{\partial}{\partial\lambda}\Big(\frac{r}{R\cos\Phi}[(a+b)\sin\beta-(a-b)\sin\chi]\Big) &= \frac{\partial}{\partial q}\Big(\frac{r}{R\cos\Phi}[(a+b)\cos\beta-(a-b)\cos\chi]\Big) \end{aligned} \tag{11}$$

According to Bugayevskiy and Snyder [3, p. 172], system (11) is referred to as the Euler–Urmaev system. It prevents the construction of a distortion-free projection of the ellipsoid onto the sphere.

If the condition is introduced:

$$a = b = 1 \tag{12}$$

then the system (11) transforms into:

$$\begin{aligned} \frac{\partial}{\partial\lambda}\left(\frac{2r}{R\cos\Phi}\cos\beta\right) &= \frac{\partial}{\partial q}\left(\frac{2r}{R\cos\Phi}\sin\beta\right), \\ -\frac{\partial}{\partial\lambda}\left(\frac{2r}{R\cos\Phi}\sin\beta\right) &= \frac{\partial}{\partial q}\left(\frac{2r}{R\cos\Phi}\cos\beta\right) \end{aligned} \tag{13}$$

This system consists of two equations with two unknown conjugate harmonic functions, which separately satisfy the Laplace equation:

$$\Delta F = \Delta\Big(\ln\frac{r}{\cos\Phi}\Big) = 0, \qquad \Delta\beta = 0 \tag{14}$$

where Δ is the Laplace operator.

At first glance, system (14), consisting of two equations with two unknown functions Φ and β, appears to permit a distortion-free projection of the ellipsoid onto the sphere. However, it should be taken into account that the isometric latitude for a sphere satisfies the condition [16, p. 6]:

$$Q = \ln\tan\Big(\frac{\pi}{4}+\frac{\Phi}{2}\Big) \tag{15}$$

This means that the first and second derivatives of the function Φ with respect to Q are:

$$\begin{aligned} \Phi_Q &= \cos\Phi, \\ \Phi_{QQ} &= -\sin\Phi\,\Phi_Q \end{aligned} \tag{16}$$

Let us determine the derivatives Φ_q and Φ_λ:

$$\begin{aligned} \Phi_q &= \Phi_Q\, Q_q, \\ \Phi_\lambda &= \Phi_Q\, Q_\lambda \end{aligned} \tag{17}$$

Differentiation of equations (17) yields:

$$\begin{aligned} \Phi_{qq} &= \Phi_{QQ}\, Q_q^2 + \Phi_Q\, Q_{qq}, \\ \Phi_{\lambda\lambda} &= \Phi_{QQ}\, Q_\lambda^2 + \Phi_Q\, Q_{\lambda\lambda} \end{aligned} \tag{18}$$

According to the first equation of system (14), the derivatives of the function $F = \ln\left(\frac{r}{\cos\Phi}\right)$ are as follows:

$$\begin{aligned} F_q &= \tau_q + \tan\Phi\,\Phi_q, \\ F_\lambda &= \tau_\lambda + \tan\Phi\,\Phi_\lambda \end{aligned} \tag{19}$$

where τ is defined as:

$$\tau = \ln r \tag{20}$$

Differentiating (19) gives:

$$\begin{aligned} F_{qq} &= \tau_{qq} + \frac{1}{\cos^2\Phi}\,\Phi_q^2 + \tan\Phi\,\Phi_{qq}, \\ F_{\lambda\lambda} &= \tau_{\lambda\lambda} + \frac{1}{\cos^2\Phi}\,\Phi_\lambda^2 + \tan\Phi\,\Phi_{\lambda\lambda} \end{aligned} \tag{21}$$

By summing the expressions in (21) and using equations (14, 16–18), the result is as follows:

$$\Delta F = \Delta\tau + \left(\frac{\Phi_Q^2}{\cos^2\Phi} + \tan\Phi\,\Phi_{QQ}\right)\left(Q_q^2 + Q_\lambda^2\right) + \tan\Phi\,\Phi_Q\left(Q_{qq} + Q_{\lambda\lambda}\right) = 0 \tag{22}$$

Taking into account that Q is a harmonic function and applying (16), the expression (22) reduces to:

$$\Delta F = \Delta\tau + \cos^2\Phi\,\left(Q_q^2 + Q_\lambda^2\right) = 0 \tag{23}$$

Substituting conditions (12) into the first two equations of system (9) gives:

$$\begin{aligned} Q_q &= \frac{r}{R\cos\Phi}\cos\beta, \\ Q_\lambda &= \frac{r}{R\cos\Phi}\sin\beta \end{aligned} \tag{24}$$

Thus, equation (23) becomes:

$$\Delta F = \Delta\tau + \frac{r^2}{R^2} = 0 \tag{25}$$

Therefore, for a distortion-free projection of a given surface onto a sphere to exist, its metric coefficient must satisfy condition (25). However, the metric coefficient of an ellipsoid of revolution does not satisfy this condition. Indeed, according to [16, pp. 5, 48], the corresponding functions are:

$$\begin{aligned} \Delta\tau &= -\frac{\cos^2\varphi\,\left(1 - e_e^2\sin^2\varphi\right)}{1 - e_e^2}, \\ r &= \frac{a_e\cos\varphi}{\sqrt{1 - e_e^2\sin^2\varphi}} \end{aligned} \tag{26}$$

where a_e and e_e^2 are the semi-major axis and the square of the eccentricity of the ellipsoid, respectively.

Figure 1(b) shows a graph illustrating the variation of the function ΔF with respect to geodetic latitude φ for the case of projecting the entire ellipsoidal surface onto a sphere. The radius of the sphere is selected such that the area under the curve where $\Delta F > 0$ equals the area where $\Delta F < 0$. That is:

$$\int_{-90^\circ}^{90^\circ} \left(\Delta\tau + \frac{r^2}{R^2}\right) d\varphi = 0 \tag{27}$$

The corresponding formula for determining the radius of the sphere is:

$$R = a_e \sqrt{\frac{8(1-e_e^2)(1-\sqrt{1-e_e^2})}{(4-e_e^2)\, e_e^2}} \tag{28}$$

For the WGS84 ellipsoid, this yields a radius of:

$$R = 6\,367\,417.740 \text{ m}.$$

The integration with respect to geodetic latitude in (27) is chosen only as a convenient meridional parameter, not as an area-preserving variable; therefore, equation (27) should not be interpreted as an optimality condition for the sphere radius. This choice allows the graph in Figure 1(b) to clearly show the regions in which the sign of the function ΔF depends on latitude.

According to condition (25), the function ΔF changes sign throughout the projection region if the radius of the sphere satisfies the following:

$$a_e\sqrt{1-e_e^2} \;<\; R \;<\; \frac{a_e}{\sqrt{1-e_e^2}} \tag{29}$$

For the WGS84 ellipsoid, the interval within which ΔF changes sign is:

$$6\,356\,752 \text{ m} < R < 6\,399\,594 \text{ m}$$

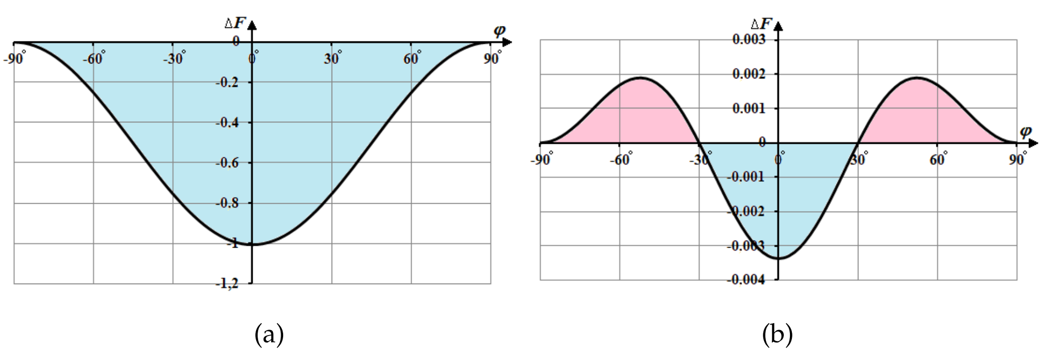


Figure 1: Graphs of the variation of the function ΔF with respect to the geodetic latitude φ: (a) mapping the surface of an ellipsoid of revolution onto a plane; (b) mapping the surface of an ellipsoid of revolution onto a sphere. The sphere radius is chosen so that the area under the curve above the horizontal axis (pink) equals the area under the curve below it (blue).

According to Figure 1(b), the function ΔF is superharmonic for latitudes $|\varphi| \leq 30^\circ$ and subharmonic for latitudes $|\varphi| \geq 30^\circ$.

For comparison, Figure 1(a) shows the corresponding function $\Delta F = \Delta\tau$ for projections onto a plane. In contrast to Figure 1(b), the function here is everywhere non-positive and superharmonic. This phenomenon was first observed in 1856 by P. Chebyshev and led to the formulation of the theorem of the best conformal projections [7, 17]. For conformal projections of an ellipsoid onto a plane, the linear scale factor is a subharmonic function that reaches its maximum at the boundary of the region. The opposite behavior of the scale, compared to that of the function ΔF, follows from the equality for conformal projections of an ellipsoid onto a plane [16, pp. 151–153]:

$$\Delta F = \Delta\tau = -\Delta \ln a,$$

according to which the properties of the functions τ and $\ln a$ (where a is the linear scale factor) are opposite: if one is superharmonic with a minimum at the boundary, the other is subharmonic with a maximum there.

Figure 1(b) shows two distinct regions within the mapping area, each with opposite harmonic properties of the extremal scale factors, which suggests that the ideal projection can be found in the set of piecewise continuous functions. A classic example of such a projection is the Markov projection [15]; [16, p. 158], which is described by piecewise continuous functions. However, this leads to a fundamental limitation: such a projection cannot be described using a single group of continuously differentiable functions. Therefore, the current work focuses on constructing the ideal projection of the ellipsoid onto a sphere within the set of continuously differentiable functions over the entire projection region.

3 Conditions defining the ideal projection of an ellipsoid onto a sphere according to the Airy criterion

The ideal projection according to the Airy criterion is defined by the condition [1]; [16, p. 55]:

$$\Xi_A^2 = \frac{1}{2\sigma} \iint_\sigma \left[(a-1)^2 + (b-1)^2\right] \, d\sigma = \min, \tag{30}$$

where $d\sigma$ is the area element of the projection surface.

For the ellipsoidal surface, the area element is given by:

$$d\sigma = r^2 \, dq \, d\lambda \tag{31}$$

Substituting (2-3, 31) into (30), the Airy condition can be rewritten as:

$$\begin{aligned} \Xi_A^2 = \frac{1}{2\sigma} \iint_\sigma \Big[& R^2 \cos^2 \Phi \, (Q_q^2 + Q_\lambda^2 + \Lambda_q^2 + \Lambda_\lambda^2) \\ & - 2rR\cos\Phi \sqrt{(Q_q + \Lambda_\lambda)^2 + (Q_\lambda - \Lambda_q)^2} \\ & + 2r^2 \Big] \, dq \, d\lambda \end{aligned} \tag{32}$$

As noted by Korn and Korn [11, p. 355], if the functional Ξ_A^2 reaches a minimum, the corresponding integrand:

$$P = R^2 \cos^2 \Phi \, (Q_q^2 + Q_\lambda^2 + \Lambda_q^2 + \Lambda_\lambda^2) - 2rR\cos\Phi \sqrt{(Q_q + \Lambda_\lambda)^2 + (Q_\lambda - \Lambda_q)^2} + 2r^2 \tag{33}$$

must satisfy the Euler–Ostrogradsky system of differential equations:

$$\begin{aligned} \frac{\partial P}{\partial Q} - \frac{\partial}{\partial q}\left(\frac{\partial P}{\partial Q_q}\right) - \frac{\partial}{\partial \lambda}\left(\frac{\partial P}{\partial Q_\lambda}\right) &= 0, \\ \frac{\partial P}{\partial \Lambda} - \frac{\partial}{\partial q}\left(\frac{\partial P}{\partial \Lambda_q}\right) - \frac{\partial}{\partial \lambda}\left(\frac{\partial P}{\partial \Lambda_\lambda}\right) &= 0 \end{aligned} \tag{34}$$

and the natural boundary conditions:

$$\begin{aligned} \frac{\partial P}{\partial Q_q}\frac{\partial \lambda}{\partial s} &= \frac{\partial P}{\partial Q_\lambda}\frac{\partial q}{\partial s}, \\ \frac{\partial P}{\partial \Lambda_q}\frac{\partial \lambda}{\partial s} &= \frac{\partial P}{\partial \Lambda_\lambda}\frac{\partial q}{\partial s} \end{aligned} \tag{35}$$

where ∂s is an arc element along the boundary of the region.

Since the functional (33) depends not only on the mapping functions Q and Λ, but also on the constant R, there is an additional condition:

$$\frac{\partial \Xi_A^2}{\partial R} = 0 \tag{36}$$

The derivatives included in system (34) are computed below. After algebraic transformations and taking equation (9) into account, the following result is obtained:

$$\begin{aligned} \frac{\partial P}{\partial Q} &= -2r^2 \sin\Phi \, (a^2 + b^2 - (a+b)), \\ \frac{\partial P}{\partial \Lambda} &= 0 \end{aligned} \tag{37}$$

$$\begin{aligned} \frac{\partial P}{\partial Q_q} &= rR\cos\Phi \left[(a+b-2)\cos\beta + (a-b)\cos\chi\right], \\ \frac{\partial P}{\partial Q_\lambda} &= rR\cos\Phi \left[(a+b-2)\sin\beta + (a-b)\sin\chi\right], \\ \frac{\partial P}{\partial \Lambda_q} &= rR\cos\Phi \left[-(a+b-2)\sin\beta + (a-b)\sin\chi\right], \\ \frac{\partial P}{\partial \Lambda_\lambda} &= rR\cos\Phi \left[(a+b-2)\cos\beta - (a-b)\cos\chi\right] \end{aligned} \tag{38}$$

Substituting equations (37–38) into equation (34) yields:

$$\begin{aligned} &\frac{\partial}{\partial q}\left[Rr\cos\Phi\big((a+b-2)\cos\beta + (a-b)\cos\chi\big)\right] \\ &\quad + \frac{\partial}{\partial \lambda}\left[Rr\cos\Phi\big((a+b-2)\sin\beta + (a-b)\sin\chi\big)\right] \\ &\quad + 2r^2\sin\Phi\big(a^2 + b^2 - (a+b)\big) = 0 \end{aligned} \tag{39}$$

$$\frac{\partial}{\partial q}\left[Rr\cos\Phi\left(-(a+b-2)\sin\beta+(a-b)\sin\chi\right)\right]$$
$$+\frac{\partial}{\partial\lambda}\left[Rr\cos\Phi\left((a+b-2)\cos\beta-(a-b)\cos\chi\right)\right]=0 \tag{40}$$

This is the Euler–Ostrogradsky system for constructing the ideal projection of the ellipsoid onto a sphere. Compared to the corresponding system for projections onto a plane (see equations (1)), this system is more complex and does not permit solutions in terms of conjugate harmonic functions.

Using (38), the natural boundary conditions (35) become:

$$\left[(a+b-2)\cos\beta+(a-b)\cos\chi\right]\frac{\partial\lambda}{\partial s}=\left[(a+b-2)\sin\beta+(a-b)\sin\chi\right]\frac{\partial q}{\partial s},$$
$$\left[-(a+b-2)\sin\beta+(a-b)\sin\chi\right]\frac{\partial\lambda}{\partial s}=\left[(a+b-2)\cos\beta-(a-b)\cos\chi\right]\frac{\partial q}{\partial s} \tag{41}$$

Dividing the first equation in (41) by the second yields:

$$(a+b-2)^2=(a-b)^2 \tag{42}$$

or equivalently:

$$|a+b-2|=a-b \tag{43}$$

Suppose the following boundary condition holds:

$$a+b-2\geq 0 \tag{44}$$

then (43) becomes:

$$b=1 \tag{45}$$

Similarly, if the boundary condition is given by:

$$a+b-2\leq 0 \tag{46}$$

then:

$$a=1 \tag{47}$$

Assuming condition (44) holds, the first equation in (41) can be written as:

$$\frac{\sin\beta+\sin\chi}{\cos\beta+\cos\chi}=\frac{\partial\lambda/\partial s}{\partial q/\partial s}$$

or simply:

$$\tan\left(\frac{\beta+\chi}{2}\right)=\frac{\partial\lambda/\partial s}{\partial q/\partial s} \tag{48}$$

As shown in [16, p. 47], for projections of an ellipsoid onto a plane, the half-sum of the angles β, χ is equal to:

$$\frac{\beta+\chi}{2}=\alpha_0 \tag{49}$$

where α_0 is the azimuth of the principal direction of the Tissot indicatrix on the original surface.

Since equalities (7) for projections of an ellipsoid onto a sphere fully coincide with the corresponding expressions for these functions in the case of projections onto a plane [16, p. 46], condition (49) holds not only for projections onto a plane but also for projections onto a sphere.

As shown in [16, p. 57], the right-hand side of equation (48) contains the tangent of the azimuth of the boundary's tangent direction of the region. Therefore, this condition can be rewritten as follows:

$$\alpha_0 = \alpha_s + \pi \cdot i, \qquad (i = 0, 1) \tag{50}$$

If inequality (46) is satisfied on the boundary, then, after algebraic transformations, equation (50) is also obtained, which indicates that it does not depend on the sign of the expression $a + b - 2$.

The obtained formulas (45, 47, 50) make it possible to formulate the boundary conditions that must be satisfied by the ideal projection of an ellipsoid onto a sphere, according to the Airy criterion. A comparison with the corresponding conditions for the ideal projection onto a plane shows that they fully coincide (see [16, p. 58]). In particular, at the boundary of the mapping area, one of the extremal linear scale factors is equal to unity, and Tissot's indicatrices are oriented either along the normals or along the tangents to the boundary of the region.

The final condition, required only for spherical projections (and absent in planar cases), is condition (36).

Differentiation of (32) with respect to R yields:

$$\begin{aligned}\frac{\partial \Xi_A^2}{\partial R} = \frac{1}{2\sigma} \iint_\sigma \Big[& 2R\cos^2\Phi\,(Q_q^2 + Q_\lambda^2 + \Lambda_q^2 + \Lambda_\lambda^2) \\ & - 2r\cos\Phi\sqrt{(Q_q + \Lambda_\lambda)^2 + (Q_\lambda - \Lambda_q)^2}\Big]\, dq\, d\lambda = 0\end{aligned} \tag{51}$$

Using equations (6, 9), equation (51) becomes:

$$\frac{2}{R}\iint_\sigma \left[a^2 + b^2 - (a+b)\right] r^2\, dq\, d\lambda = 0$$

or, discarding the condition $R \to \infty$:

$$\iint_\sigma \left[a^2 + b^2 - (a+b)\right] r^2\, dq\, d\lambda = 0 \tag{52}$$

If the integrand is not identically zero, then to ensure the integral equals zero over the entire region, it must take positive values in some parts and negative in others. Assuming the distortions are of the first order of smallness, the expression (52) becomes:

$$\iint_\sigma (a + b - 2) r^2\, dq\, d\lambda \approx 0 \tag{53}$$

According to [3, p. 27], the area distortion can be written as follows:

$$V_p = p - 1 = ab - 1$$

where p is the area scale factor.

As shown in [3, p. 28], the area distortion, accurate to second-order terms, is given by:

$$V_p \approx V_a + V_b \tag{54}$$

where $V_a = a - 1,\ V_b = b - 1$ are the maximum and minimum linear distortions.

Substituting the maximum and minimum scale factors gives [16, p. 32]:

$$V_p \approx a + b - 2$$

Therefore, it can be stated that, for the ideal Airy projection of an ellipsoid onto a sphere, the integral of area distortion over the entire mapped region is equal to zero. Moreover, the portion of the region with negative area distortion is equal in size to the portion with positive area distortion.

4 Ideal Airy projection of the surface of an ellipsoid of revolution bounded by a parallel

Consider the case where the projection region is a portion of the ellipsoidal surface centered at the pole and bounded by a parallel with isometric latitude q_0. The relationship between coordinates on the ellipsoid and those on the sphere is defined as:

$$\Lambda = \lambda, \qquad Q = f(q) \tag{55}$$

Substituting (55) into equation (32), with consideration of (6, 9), and applying standard transformations yields:

$$\Xi_A^2 = \frac{\pi}{\sigma} \int_{q_0}^{90^\circ} \left[R^2 \cos^2 \Phi \, (Q_q^2 + 1) - 2rR \cos \Phi \, (Q_q + 1) + 2r^2 \right] dq \tag{56}$$

The relationship between the isometric latitude and geodetic latitude is given by [8, p. 64]:

$$q_\varphi = \frac{M}{r} \tag{57}$$

where r, M are the radius of the parallel and the radius of curvature of the meridian, respectively.

The derivative relationship between Φ_φ and Q_q becomes:

$$\Phi_\varphi = \Phi_Q Q_\varphi = \Phi_Q Q_q q_\varphi \tag{58}$$

or taking into account (16, 57):

$$Q_q = \Phi_\varphi \, \frac{r}{M \cos \Phi} \tag{59}$$

Substitution of equation (59) into equation (56) yields the following integral:

$$\Xi_A^2 = \frac{\pi}{\sigma} \int_{\varphi_0}^{90^\circ} \left[\frac{R^2 r}{M} \, \Phi_\varphi^2 - 2rR \, \Phi_\varphi + \frac{R^2 M}{r} \cos^2 \Phi - 2RM \cos \Phi + 2rM \right] d\varphi \tag{60}$$

where φ_0 is the geodetic latitude of the boundary parallel corresponding to q_0.

Under the assumption of a projection with an orthogonal map graticule, the extremal linear scale factors satisfy [3, p. 16]; [16, p. 31]:

$$a = \max(m, n), \qquad b = \min(m, n) \tag{61}$$

Given the definition of the projection in (55), the extremal scale factors a, b correspond to scale factors along meridians and parallels, respectively:

$$m = \frac{R\Phi_\varphi}{M}, \qquad n = \frac{R\cos\Phi}{r} \tag{62}$$

With these definitions, equation (60) takes the form:

$$\Xi_A^2 = \frac{\pi}{\sigma}\int_{\varphi_0}^{90^\circ} \left[(m-1)^2 + (n-1)^2\right] rM \, d\varphi$$

Assuming that the spherical latitude Φ is close to the geodetic latitude φ, the function Φ can be represented as follows:

$$\Phi = \varphi + x \tag{63}$$

where x contains terms no greater than first-order small quantities [16, p. 32].

The corresponding functions $\cos\Phi$ and $\cos^2\Phi$ can be approximated using [4, p. 79]:

$$\cos\Phi = \cos(\varphi + x) \approx \cos\varphi - x\sin\varphi - \tfrac{1}{2}x^2\cos\varphi + \dots \tag{64}$$

$$\cos^2\Phi \approx \cos^2\varphi - x\sin 2\varphi - x^2\cos 2\varphi + \dots \tag{65}$$

Differentiating (63) with respect to φ gives:

$$\Phi_\varphi = 1 + x_\varphi \tag{66}$$

Substituting (64-66) into (60), the integral becomes:

$$\begin{aligned} \Xi_A^2 \approx \frac{\pi}{\sigma}\int_{\varphi_0}^{90^\circ} \Big[&\frac{R^2 r}{M}x_\varphi^2 - \frac{R^2 M}{r}x^2\cos 2\varphi + RMx^2\cos\varphi + \frac{2R^2 r}{M}x_\varphi \\ &- 2rRx_\varphi - \frac{R^2 M}{r}x\sin 2\varphi + 2RMx\sin\varphi + \frac{R^2 r}{M} \\ &- 2rR + \frac{R^2 M}{r}\cos^2\varphi - 2RM\cos\varphi + 2rM\Big]\, d\varphi \end{aligned} \tag{67}$$

The function x is sought in the form of a trigonometric polynomial:

$$x = \sum_{i=1}^{N}\left(a_{2i-1}\cos((2i-1)\varphi) + b_{2i}\sin(2i\varphi)\right) \tag{68}$$

where $a_{2i-1},\ b_{2i}$ are constants.

In formula (68), not the full set of trigonometric functions is used, but only those that are divisible by $\cos\varphi$ without resulting in a singularity. This choice is due to the presence of

$\cos\varphi$ in the denominator of one of the terms in the integrand of equation (67), specifically in the expression $\dfrac{R^2 M}{r}(x^2\cos 2\varphi)$. Neglecting this feature may lead to a solution for x that tends to infinity as $\varphi = \pm 90°$.

Differentiating (68) with respect to φ yields:

$$\begin{aligned} x_\varphi &= \sum_{i=1}^{I}\left(-(2i-1)a_{2i-1}\sin((2i-1)\varphi) + 2i\,b_{2i}\cos(2i\varphi)\right), \\ x_{\varphi\varphi} &= \sum_{i=1}^{I}\left(-(2i-1)^2 a_{2i-1}\cos((2i-1)\varphi) - (2i)^2 b_{2i}\sin(2i\varphi)\right) \end{aligned} \tag{69}$$

To find the minimum of the functional (67), one may apply methods from the calculus of variations [11, p. 350]. However, a simpler approach in this case is to take into account that the functional depends on the parameters a_{2i-1}, b_{2i}, where $i = 1, 2, \ldots, N$. If the functional attains a minimum with respect to these parameters, they can be determined from the following system of equations:

$$\begin{aligned} &\frac{\partial \Xi_A^2}{\partial R} = 0, \\ &\frac{\partial \Xi_A^2}{\partial a_{2i-1}} = 0, \qquad \frac{\partial \Xi_A^2}{\partial b_{2i}} = 0, \qquad (i = 1, 2, \ldots, N) \end{aligned} \tag{70}$$

The first condition in (70) yields the formula for the radius of the sphere:

$$R\int_{\varphi_0}^{90°}\left(\frac{r}{M}\Phi_\varphi^2 + \frac{M}{r}\cos^2\Phi\right)d\varphi = \int_{\varphi_0}^{90°}\left(r\Phi_\varphi + M\cos\Phi\right)d\varphi \tag{71}$$

The remaining $2N$ conditions form a system of $2N$ equations that is linear with respect to the parameters a_{2i-1}, b_{2i}, and can be written as follows:

$$\sum_{j=1}^{N}\left(A_{i,j}\,a_{2j-1} + B_{i,j}\,b_{2j}\right) - C_i = 0, \qquad i = 1, 2, \ldots, 2N \tag{72}$$

In equations (72), the coefficients $A_{i,j}$, $B_{i,j}$, and C_i are definite integrals. To compute them, the working formulas provided in Appendix A were used. All integrals were evaluated approximately using Simpson's rule.

The formulas for the integrals $A_{i,j}$, $B_{i,j}$, and C_i include the quantity a_e/R (see Appendix A), which can be determined from equation (71) using the previously obtained values of the parameters a_{2i-1}, b_{2i}. Since the surface of the sphere realizing the ideal projection is assumed to be close to that of the ellipsoid, the ratio a_e/R can be set to unity in the first approximation. The problem is solved using the method of successive approximations. These formulas were used to construct the ideal projection of the northern hemisphere, bounded by the equator:

$$\varphi_0 = 0 \tag{73}$$

The results of solving system (71)–(72) under condition (73) with $N = 4$ for the WGS84 ellipsoid are presented in Table 1. The extremal distortions and the Airy criterion value for the resulting projection are given in Table 2. Figure 2(a) illustrates the variation of extremal linear distortions with respect to latitude. The horizontal axis is reversed to display distortion values from the central point (the pole) to the boundary (the equator).

Despite the fact that the condition of setting one of the extremal distortions to zero was not used when determining the parameters of the ideal projection (see equations (45) and (47)), Figure 2(a) shows that the minimum distortion V_b at the equator is indeed zero. The maximum linear distortion V_a at the equator is positive and equal to 0.054%. At the center of the region, both extremal linear distortions are positive and equal to 0.025%.

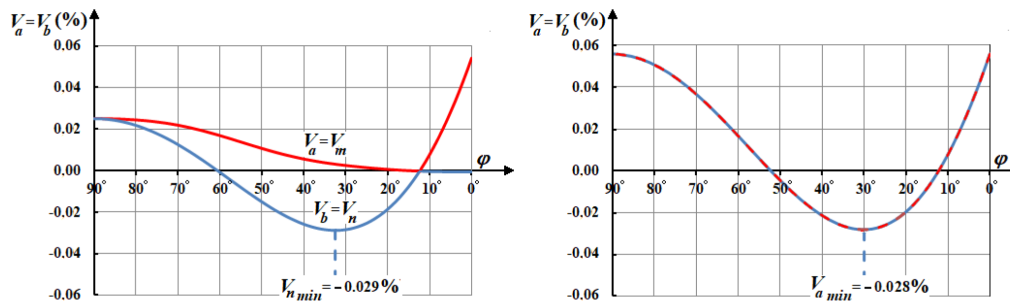
Parameter	Projection of a semi-ellipsoid	Projection of the entire ellipsoid
a_1	$3.942\,779\,976\,784\,07 \times 10^{-3}$	0
a_3	$-7.301\,856\,416\,545\,50 \times 10^{-4}$	0
a_5	$1.192\,201\,852\,907\,86 \times 10^{-4}$	0
a_7	$-7.021\,290\,142\,989\,11 \times 10^{-6}$	0
b_2	$-4.168\,504\,919\,549\,37 \times 10^{-3}$	$-2.684\,377\,642\,155\,86 \times 10^{-3}$
b_4	$3.252\,536\,556\,054\,44 \times 10^{-4}$	$2.786\,169\,372\,808\,09 \times 10^{-6}$
b_6	$-3.452\,386\,796\,133\,45 \times 10^{-5}$	$-2.775\,755\,035\,935\,09 \times 10^{-9}$
b_8	$7.668\,961\,942\,573\,96 \times 10^{-7}$	$-6.571\,253\,729\,500\,93 \times 10^{-10}$
$\frac{a_e}{R}$	$9.994\,534\,945\,914\,11 \times 10^{-1}$	$1.001\,119\,551\,298\,01$

Table 1: Parameters of the function x and the value a_e/R of ideal projections of the ellipsoid onto the sphere according to the Airy criterion.

Characteristic	Semi-ellipsoid projection		Entire ellipsoid projection	
	ideal	conformal	ideal	conformal
Ξ (%)	0.017	0.025	0.063	0.100
R (m)	6381624.592	6381731.102	6371004.334	6371003.998
Extremal distortions at equator				
V_b (%)	0	0.056	-0.112	-0.112
V_a (%)	0.054	0.056	0.022	-0.112
V_p (%)	0.054	0.112	-0.089	-0.224
ω (arcmin)	1.881	0	4.623	0
Extremal distortions at the pole				
V_a (%)	0.025	0.056	0.089	0.223
V_b (%)	0.025	0.056	0.089	0.223
V_p (%)	0.050	0.112	0.178	0.446
ω (arcmin)	0	0	0	0

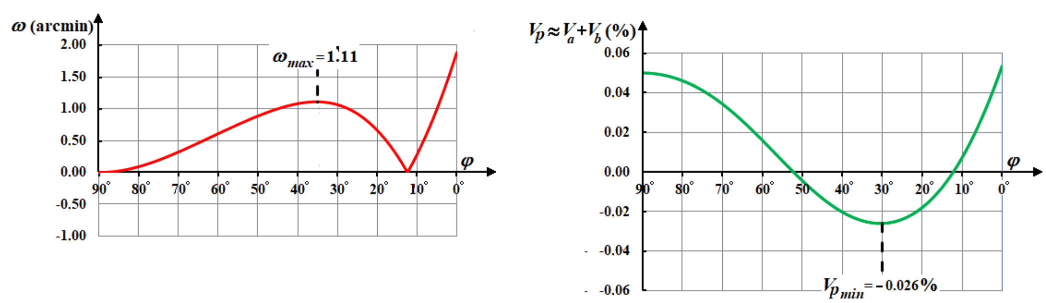
Table 2: Extremal distortions for ideal and conformal projections of the semi-ellipsoid and entire ellipsoid.

For the same mapping area, i.e., a semi-ellipsoid, the best Airy conformal projection was constructed. The formulas used to compute this projection are presented in Appendix B. According to the data in Table 2, the maximum linear distortion of the best conformal projection is 0.056%, which is nearly equal to that of the ideal projection (0.054%). At the center of the mapping area, at the point with latitude $\varphi = 90°$, the maximum linear distortion of the best conformal projection is also 0.056%, which is more than twice the value for the ideal projection (0.025%). The value of the Airy criterion for the ideal projection is 0.017%, which is 0.008% lower than that of the best conformal projection (0.025%).



(a) (b)

Figure 2: Graphs of extremal linear distortions for the projection of a semi-ellipsoid onto a sphere: (a) ideal projection according to the Airy criterion; (b) best conformal projection (Airy criterion).



(a) (b)

Figure 3: Graphs of distortions for the ideal Airy projection of a semi-ellipsoid onto a sphere: (a) extremal angular distortions; (b) area distortions.

Figure 3 shows the graphs of maximum angular distortions and area distortions for the ideal projection. A comparison of the graphs in Figure 2(b) and Figure 3(b) shows that the linear distortions of the best conformal projection and the area distortions of the ideal Airy projection are nearly identical. A similar conclusion was previously made in [16, pp. 70–74] for projections of an ellipsoid onto a plane. Unlike the conformal projection, the ideal projection according to the Airy criterion introduces angular distortions, with a maximum value of 1.88 arcminutes.

Figure 2(b) shows the graph of the function:

$$V_p \approx V_a + V_b \approx a^2 + b^2 - (a + b)$$

according to which part of the function is positive and part is negative. As noted earlier, the integral of this function over the mapping area must be equal to zero, accurate to second-order terms (see equation (52)). As a result of the calculations, the following value was obtained:

$$\int_0^{90^\circ} (V_a + V_b) M r \, d\varphi \, d\lambda \approx -15 \, \text{km}^2$$

Given that the surface area of the WGS84 semi-ellipsoid is $2.6 \times 10^8 \, \text{km}^2$, it can be concluded that equation (52) is satisfied for the resulting projection with a relative accuracy of 6×10^{-8}.

5 Ideal Airy projection of the entire surface of an ellipsoid of revolution onto a sphere

The ideal Airy projection of the entire ellipsoidal surface can be constructed using the same approach described in the previous section. Since the equator of the ellipsoid must coincide with that of the sphere, the function x defined in system (72) must be odd:

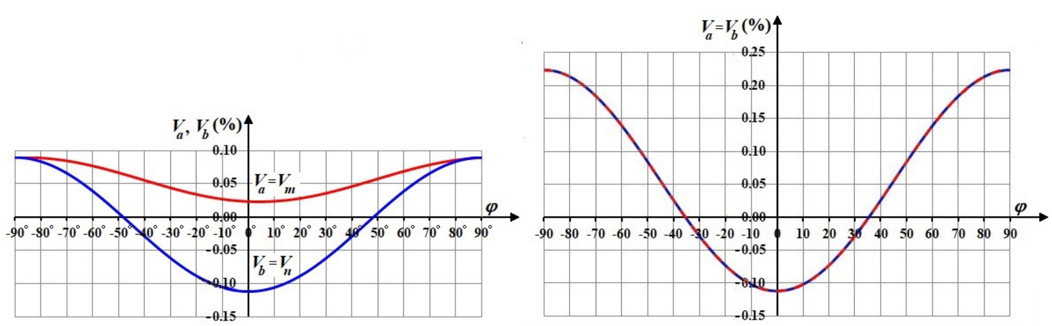
$$a_{(2i-1)} = 0, \quad (i = 1, 2, \ldots, N) \tag{74}$$

To determine the constants b_{2i}, the same system of equations (71–72) is used, but now the integration is performed over the entire interval:

$$-90^\circ \leq \varphi \leq 90^\circ$$

The numerical values of the coefficients b_{2i} are provided in Table 1, while the key projection characteristics are summarized in Table 2.

Figure 4 shows the graphs of extremal linear distortions for: (a) the ideal Airy projection; (b) the best conformal projection according to the Airy criterion, constructed for the entire surface of the ellipsoid.



(a) (b)

Figure 4: Graphs of extremal linear distortions for projections of an ellipsoid onto a sphere: (a) the ideal Airy projection; (b) the best conformal projection according to the Airy criterion.

According to Table 2 and Figure 4(a), the maximum linear distortion for the ideal projection—reached at latitudes $\pm 90°$—is 0.089%, and it applies to both extremal distortions V_a and V_b. At the equator, the maximum distortion is 0.022%, and the minimum reaches -0.112%.

A comparison of the ideal projection with the best conformal Airy projection, both constructed for the entire surface of the ellipsoid, shows that the minimum linear distortion at the equator, equal to –0.112%, is the same for both projections. However, the maximum linear distortion at the poles in the conformal projection (0.223%) is two and a half times greater than that in the ideal projection (0.089%).

Thus, the distortions of lengths and areas in ideal projections are smaller than the corresponding distortions in conformal projections. However, this comes at the cost of introducing angular distortions. In the ideal projection of a semi-ellipsoid onto a sphere, the maximum angular distortion at the equator is 1.88 arcminutes. For the projection of the entire ellipsoid onto a sphere, the maximum angular distortion at the equator increases to 4.62 arcminutes.

6 Conclusions

The projections of the ellipsoid onto a sphere presented in this paper represent the best possible mappings—in the set of continuously differentiable functions—according to the Airy criterion. As described by Bugayevskiy and Snyder [3, p. 193], such mappings are referred to as ideal with respect to the selected distortion criterion.

These ideal projections minimize linear distortion and optimize its distribution throughout the projection region. Compared to conformal projections, they reduce both linear and area distortions, albeit at the expense of introducing small angular distortions.

The constructed ideal projections have potential applications in geodesy (e.g., for solving direct and inverse geodetic problems) and in cartography, particularly for use in double projection techniques, where an intermediate projection onto a sphere is employed.

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Appendices

Appendix A. Working formulas for calculating definite integrals $A_{i,j}$, $B_{i,j}$, C_i

The following notations are introduced:

$$\begin{aligned} d_1 &= \frac{r}{M} = \frac{\cos\varphi\,\left(1-e_e^2\sin^2\varphi\right)}{1-e_e^2}, \\ d_2 &= \frac{r}{R} = \frac{a_e}{R}\,\frac{\cos\varphi}{\sqrt{1-e_e^2\sin^2\varphi}}, \\ d_3 &= \frac{M\cos\varphi}{r} = \frac{1-e_e^2}{1-e_e^2\sin^2\varphi}, \\ d_4 &= \frac{M}{R} = \frac{a_e}{R}\,\frac{1-e_e^2}{\left(1-e_e^2\sin^2\varphi\right)^{3/2}} \end{aligned} \tag{A.1}$$

For $i = 1,\dots,N$, $j = 1,\dots,N$ the definite integrals $A_{i,j}$, $B_{i,j}$, C_i are given by:

$$\begin{aligned} A_{i,j} = \int_{\phi_0}^{90^\circ} \Big[& d_1(2i-1)(2j-1)\,\sin\big((2i-1)\varphi\big)\,\sin\big((2j-1)\varphi\big) \\ & - d_3\,\frac{\cos\big((2i-1)\varphi\big)}{\cos\varphi}\,\cos\big((2j-1)\varphi\big)\,\cos 2\varphi \\ & + d_4\,\cos\big((2i-1)\varphi\big)\,\cos\big((2j-1)\varphi\big)\,\cos\varphi\Big]\,d\varphi \end{aligned} \tag{A.2}$$

$$\begin{aligned} B_{i,j} = \int_{\phi_0}^{90^\circ} \Big[& -\,2d_1(2i-1)j\,\sin\big((2i-1)\varphi\big)\,\cos\big(2j\varphi\big) \\ & -\,d_3\,\frac{\cos\big((2i-1)\varphi\big)}{\cos\varphi}\,\sin\big(2j\varphi\big)\,\cos 2\varphi \\ & +\,d_4\,\cos\big((2i-1)\varphi\big)\,\sin\big(2j\varphi\big)\,\cos\varphi\Big]\,d\varphi \end{aligned} \tag{A.3}$$

$$C_i = \int_{\phi_0}^{90^\circ} \left[(d_1-d_2)\cdot(2i-1)\sin\left((2i-1)\varphi\right) + (d_3-d_4)\cos\left((2i-1)\varphi\right)\sin\varphi\right] d\varphi \tag{A.4}$$

For $i = N+1,\dots,2N$, $j = 1,\dots,N$ the definite integrals $A_{i,j}$, $B_{i,j}$, C_i are defined as follows:

$$\begin{aligned} A_{i,j} = \int_{\phi_0}^{90^\circ} \Big[& -\,2d_1(i-N)(2j-1)\,\sin\big(2(i-N)\varphi\big)\,\sin\big((2j-1)\varphi\big) \\ & -\,d_3\,\frac{\sin\big(2(i-N)\varphi\big)}{\cos\varphi}\,\cos\big((2j-1)\varphi\big)\,\cos 2\varphi \\ & +\,d_4\,\sin\big(2(i-N)\varphi\big)\,\cos\big((2j-1)\varphi\big)\,\cos\varphi\Big]\,d\varphi \end{aligned} \tag{A.5}$$

$$B_{i,j} = \int_{\phi_0}^{90^\circ} \Big[4d_1(i-N)j\ \cos\big(2(i-N)\varphi\big)\ \cos\big(2j\varphi\big) - d_3 \frac{\sin\big(2(i-N)\varphi\big)}{\cos\varphi}\ \sin\big(2j\varphi\big)\ \cos 2\varphi + d_4\ \sin\big(2(i-N)\varphi\big)\ \sin\big(2j\varphi\big)\ \cos\varphi \Big]\, d\varphi \tag{A.6}$$

$$C_i = \int_{\phi_0}^{90^\circ} \left[-2(d_1-d_2)\cdot(i-N)\cos\left(2(i-N)\varphi\right) + (d_3-d_4)\sin\left(2(i-N)\varphi\right)\sin\varphi\right] d\varphi \tag{A.7}$$

Appendix B. The best conformal projection of an ellipsoid onto a sphere according to the Airy criterion for a region bounded by the parallel of latitude φ_0

The condition for conformality in projections of an ellipsoid onto a sphere is expressed as follows [3, p. 33]:

$$m = n \tag{B.1}$$

or

$$\frac{R\,\Phi_\varphi}{M} = \frac{R\cos\Phi}{r} \tag{B.2}$$

The value of the Airy criterion in this case is given by:

$$\Xi^2 = \frac{2\pi}{\sigma}\int_{\varphi_0}^{90^\circ}\left[\left(\frac{R\cos\Phi}{r} - 1\right)^2\right] Mr\,d\varphi. \tag{B.3}$$

Integrating equation (B.2) yields a relationship between the latitude Φ of a point on the sphere and the corresponding latitude φ of a point on the ellipsoid [3, p. 33]:

$$\tan\left(\frac{\pi}{4} + \frac{\Phi}{2}\right) = kU \tag{B.4}$$

where:

$$U = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\left(\frac{1 - e_e\sin\varphi}{1 + e_e\sin\varphi}\right)^{\frac{e_e}{2}} \tag{B.5}$$

Using the notation in (B.4), the linear scale factor of the conformal projection of the ellipsoid onto the sphere is given by:

$$m = n = \frac{R\cos\Phi}{r} = \frac{R}{r}\left(\frac{2kU}{1 + k^2U^2}\right) \tag{B.6}$$

In this case, the condition for determining the best conformal projection according to the Airy criterion (see equation (30)) can be written as:

$$\Xi^2 = \frac{2\pi}{\sigma}\int_{\phi_0}^{90^\circ}\left(\frac{R}{r}\left(\frac{2kU}{1 + k^2U^2}\right) - 1\right)^2 Mr\,d\varphi = \min \tag{B.7}$$

The functional (B.7) depends on two constants R and k. If some values of the constants minimize the functional Ξ^2, then they satisfy the following system of equalities:

$$\frac{\partial \Xi^2}{\partial R} = 0, \quad \frac{\partial \Xi^2}{\partial k} = 0 \tag{B.8}$$

Differentiating equation (B.7) with respect to R, k yields:

$$\frac{\partial \Xi^2}{\partial R} = \frac{2\pi}{\sigma}\int_{\phi_0}^{90^\circ}\left[2\frac{R}{r}M\left(\frac{4k^2U^2}{(1+k^2U^2)^2}\right) - M\left(\frac{4kU}{(1+k^2U^2)}\right)\right]d\varphi = 0 \tag{B.9}$$

$$\frac{\partial \Xi^2}{\partial k} = \frac{2\pi R}{\sigma}\int_{\phi_0}^{90^\circ}\left[\frac{R}{r}M\left(8kU^2\frac{(1-k^2U^2)}{(1+k^2U^2)^3}\right) - M\left(\frac{4U(1-k^2U^2)}{(1+k^2U^2)^2}\right)\right]d\varphi = 0 \tag{B.10}$$

Eliminating the constant R from equations (B.9-B.10) yields the following equation for determining the constant k:

$$\begin{aligned}&\left(\int_{\phi_0}^{90^\circ}\frac{M}{r}\frac{U^2(1-k^2U^2)}{(1+k^2U^2)^3}\,d\varphi\right)\cdot\left(\int_{\phi_0}^{90^\circ}\frac{MU}{1+k^2U^2}\,d\varphi\right)\\ &= \left(\int_{\phi_0}^{90^\circ}\frac{M}{r}\frac{U^2}{(1+k^2U^2)^2}\,d\varphi\right)\cdot\left(\int_{\phi_0}^{90^\circ}\frac{MU(1-k^2U^2)}{(1+k^2U^2)^2}\,d\varphi\right)\end{aligned} \tag{B.11}$$

Equation (B.11) can be solved by the method of successive approximations.

The radius of the sphere that best fits this projection problem can be found using one of the following formulas:

$$R = \frac{1}{2k}\frac{\int_{\phi_0}^{90^\circ}\left[M\left(\frac{U(1-k^2U^2)}{(1+k^2U^2)^2}\right)\right]d\varphi}{\int_{\phi_0}^{90^\circ}\left[\frac{M}{r}\left(U^2\frac{(1-k^2U^2)}{(1+k^2U^2)^3}\right)\right]d\varphi} = \frac{1}{2k}\frac{\int_{\phi_0}^{90^\circ}\left[M\left(\frac{U}{(1+k^2U^2)}\right)\right]d\varphi}{\int_{\phi_0}^{90^\circ}\left[\frac{M}{r}\left(\frac{U^2}{(1+k^2U^2)^2}\right)\right]d\varphi} \tag{B.12}$$

The parameters of the best conformal projection of a semi-ellipsoid onto a sphere are listed in Table B.1.

Parameter	Semi-ellipsoid projection	Entire ellipsoid projection
k	1.00336371415339	1
R (m)	6381731.102	6371003.997

Table 3: Constants of the best conformal projections of a semi-ellipsoid and the entire ellipsoid onto a sphere.

The extremal linear distortions and the Airy criterion values of the resulting projections are presented in Table 2.

When the entire surface of an ellipsoid is projected onto a sphere, the equator of the ellipsoid must be mapped onto the equator of the sphere. In this case, the parameter k is equal to 1 (see equations (B.4)–(B.5)). The radius can then be determined using the following formula:

$$R = \frac{1}{2}\frac{\int_{-90^\circ}^{90^\circ}M\frac{U}{1+U^2}\,d\varphi}{\int_{-90^\circ}^{90^\circ}\frac{M}{r}\frac{U^2}{(1+U^2)^2}\,d\varphi} \tag{B.13}$$